Week 5

Readings: Chapter 3 of Yang.
Problems:

1. Yang: problems 3.14.3 (and 3.14.11, optional)

2. Matthieu equation:

$$\dot{q} = p \quad (1)$$
$$\dot{p} = -(1+\epsilon\cos(\omega t))q \quad (2)$$

Give an exact solution of this equation using Dyson T-ordered exponential. Investigate numerical approximations of this formula and compare with numerical solutions with *NIntegrate* and second order Runge-Kutta (in C++). Determine the region of instability in the $\omega - \epsilon$ plane.

3. The Double Pendulum (suggested language: Mathematica)

   (a) Starting with the Lagrangian

$$L = \frac{1}{2}(m_1+m_2)l_1^2\dot{\phi_1}^2+\frac{1}{2}m_2l_2^2(\dot{\phi_1}+\dot{\phi_2})^2+m_2l_1l_2\dot{\phi_1}(\dot{\phi_1}+\dot{\phi_2})\cos(\phi_2)+(m_1+m_2)gl_1\cos(\phi_1)+m_2gl_2\cos(\phi_2+\phi 1)\ , \quad (3)$$

   show that the The hamiltonian for the double pendulum is given as

$$H = T + V \quad (4)$$

$$T = \frac{1}{2\Delta}(Ap_1^2 - 2Bp_1p_2 + Cp_2^2)$$
$$A = m_2l_2^2$$
$$B = m_2l_2^2 + m_2l_1l_2Cos(\phi_2)$$
$$C = (m_1+m_2)l_1^2 + m_2l_2^2 + 2m_2l_1l_2Cos(\phi_2)$$
$$\Delta = l_1^2l_2^2(m_2^2(Sin(\phi_2))^2 + m_1m_2) \quad (5)$$

$$V = -g((m_1+m_2)l_1Cos(\phi_1) + m_2l_2Cos(\phi_1+\phi_2))\ . \quad (6)$$

   (Note: ϕ_2 is the angle between the direction of the first pendulum and the second pendulum). This can be shown by noticing that the kinetic energy is a quadratic form in $\dot{\phi_1}$ and $\dot{\phi_2}$. It is then easy to show that the corresponding part in the Hamiltonian is a quadratic form in p_1 and p_2 with a matrix that is the inverse of the one appearing in the Lagrangian.

   (b) Write the Hamilton equations of motion and solve numerically for simple initial conditions of your choice.

   (c) Enumerate all the singular points (where rhs of Hamilton equations are all zero; they can be found without solving complicated transcendental equations). They have a very simple physical interpretation.

(d) Describe the general shape of $V(\phi_1, \phi_2)$. Compare the location of the extrema with the location of the singular points in the (ϕ_1, ϕ_2) plane.

(e) Check analytically that in the limit $m_2 << m_1$, the first pendulum behaves approximately like a single planar pendulum (example discussed in class). Integrate the ODE numerically for values of the arbitrary parameters of your choice but such that $m_2 << m_1$ and describe the motion in the (ϕ_1, p_1) plane.

(f) investigate numerically and analytically the linear behavior (small oscillations) near the stable equilibrium position.