The Double Pendulum

Step 1: Checking the numerical integration with simple examples

The hamiltonian for the double pendulum is given as

$$H = T + V \tag{1}$$

with

$$\begin{aligned} T &= \frac{1}{2\Delta}(Ap_1^2 - 2Bp_1p_2 + Cp_2^2) \\ A &= m_2 l_2^2 \\ B &= m_2 l_2^2 + m_2 l_1 l_2 Cos(\phi_2) \\ C &= (m_1 + m_2) l_1^2 + m_2 l_2^2 + 2 m_2 l_1 l_2 Cos(\phi_2) \\ \Delta &= l_1^2 l_2^2 (m_2^2 (Sin(\phi_2))^2 + m_1 m_2) \end{aligned} \tag{2}$$

and

$$V = -g((m_1 + m_2) l_1 Cos(\phi_1) + m_2 l_2 Cos(\phi_1 + \phi_2)) \ . \tag{3}$$

1. Write the Hamilton equations of motion (4 first order ODE, they are somehow long, but you will manipulate them with Mathematica later)

2. Enumerate all the singular points (they can be found without solving complicated transcendental equations). They have a very simple physical interpretation.

3. Describe the general shape of $V(\phi_1, \phi_2)$. Compare the location of the extrema with the location of the singular points in the (ϕ_1, ϕ_2) plane.

4. Check analytically that in the limit $m_2 << m_1$, the first pendulum behaves approximately like a single planar pendulum (example discussed in class). Integrate the ODE numerically for values of the arbitrary parameters of your choice but such that $m_2 << m_1$ and describe the motion in the (ϕ_1, p_1) plane.

Step 2: Linear behavior

Consider the ODE for the hamiltonian of the double pendulum given in "step 1", with $m_1 = m_2 = 1$, $l_1 = l_2 = 1$ and $g = 1$.

1. Consider the singular point $\phi_1 = 0$ and $\phi_2 = 0$ which is a stable equilibrium point. Write the linearized ODE near this singular point.

2. Calculate the eigenvalues and the eigenvectors of the matrix associated with the linearized ODE.

3. Using the eigenvectors, give a brief and concrete description of the small vibrations (there are two "normal modes" with distinct periods of oscillations) near the equilibrium.

4. What are the two periods of oscillations? Is their ratio a rational number? What can you conclude from that?

5. Consider initial values corresponding to the mode having the smallest period of oscillation with $p_1 = p_2 = 0$. This amounts the take the initial values of the angles in a definite ratio, however the overall scale is not fixed. Integrate the original ODE (not the linearized one) numerically as in step 1. Take the initial angles sufficiently small to obtain results in good agreement with the linearized equations. In particular you should check that the period corresponds to the one calculated above. Describe the projection of the motion in the (ϕ_1, ϕ_2), (p_1, p_2) and (ϕ_1, p_1) planes.

6. Compare these projections with the ones obtained by taking initial values of the angles of the same (small) order of magnitude but not in the particular ratio chosen above.

7. Take the initial values as in the previous case and increase the overall scale until sizable departure from the linear behavior can be observed. Describe the projection of the motion in the (ϕ_1, ϕ_2), (p_1, p_2) and (ϕ_1, p_1) planes.

8. Consider the singular point $\phi_1 = 0$ and $\phi_2 = \pi$ which is in part stable and in part unstable. Write the linearized ODE near this singular point. Calculate the eigenvalues and the eigenvectors of the matrix associated with the linearized ODE. Consider initial values corresponding to the stable mode, with $p_1 = p_2 = 0$. Take small values of the overall factor (as above) in order to obtain results in good agreement with the linearized equations, during one period, when you integrate numerically the original ODE. For how many periods are you able to maintain this agreement? Interpret quantitatively your result (in terms of the real eigenvalue associated with the unstable direction).

Step 3: Non-Linear behavior

Consider the ODE for the hamiltonian of the double pendulum given in "step 1", with $m_1 = m_2 = 1$, $l_1 = l_2 = 1$ and $g = 1$. Take initial values such that $\phi_2 = 0$ and $\dot{\phi}_2 > 0$. This conditions together with a given choice for E, the total energy, defines a *surface* in the four dimensional phase space. In the following we study the map of this surface into itself obtained by picking a point of the surface as our initial values and evolving it with the equations of motion until it returns to the surface. This "return map" also called Poincare section allows one to detect the presence of additional constant of motion. General warnings: ϕ_1 and ϕ_2 are periodic variables: ϕ_1 and $\phi_1 + 2\pi$ are the very same thing; characteristic times vary hugely with the energy, so always start with "short" evolutions to get a first idea; check that the energy is conserved and that your results are numerically stable.

1. Describe the projection in the (ϕ_1, p_1) plane of the surface defined above, for $E = -2.9, 0, 100$. (This is a very simple question, it amounts to solve a quadratic equation).

2. Pick five points of your choice on the surfaces described in 1. (take the points as far apart as possible from each others) for each of the three energies $E = -3, 0, 100$. Study the return map for times which are long enough to distinguish among the following possibilities: periodicity, quasiperiodicity and ergodicity. Describe the maps for each energies.

3. Compare the evolution of one of your pick at $E = 100$ with the evolution for the Hamiltonian with $g = 0$. Explain your findings from the point of view of the equations of motion. Hint: when $g = 0$, H is independent of ϕ_1.

4. Repeat questions 1. and 2. for an energy E of your choice such that $-2 < E < 3$. Report your more surprising findings.