29:273 (Spring 2007)

Readings and Homework

D=Dirac, KT=Kolb and Turner, HEL: Hobson, Efstathiou and Lasenby.

Week 1

Readings: D, Chs. 1-3 ; KT: Ch. 1 (for inspiration); Review special relativity (Feynman Lectures or Jackson). See also: HEL, Ch. 1.

Week 2

Readings: D, Chs. 4-9. See also: HEL, Chs. 2 and 3.

Problem set 1 (due Tuesday 1/30)

1. Calculate the torque on Eötvös torsion balance described in class. Explain how the gravitational mass and the inertial mass enter in the equation.
2. Enumerate all the real 2 by 2 matrices M such that $M^T M = 1$ (1 being the identity matrix). Show that they form a group.
3. Using G, M_{Sun} and c, form a quantity that has the units of a length and calculate it in meters. Same question but with the mass of a basketball and the mass of an electron.
4. Using G, $\hbar$ and c, form quantities that have the units of length, time and mass and calculate them in MKS units.
5. Calculate the Christoffel symbols (of both kinds) for the sphere (in spherical coordinates). There is no time component in this problem, it is just a surface embedded in a 3 dimensional Cartesian space.

Week 3

Readings: D, Chs. 10-14. See also: HEL, Ch. 7.

Problem set 2 (due Tuesday 2/6)

1. Show that under a general coordinate transformation, a symmetric rank 2 tensor remains symmetric.
2. Given that $\Gamma_{\mu\nu\sigma} = y^n_{,\mu} y_{n,\nu,\sigma}$, show that $\Gamma_{\mu\nu\sigma} = \frac{1}{2}(g_{\mu\nu,\sigma} + g_{\mu\sigma,\nu} - g_{\nu\sigma,\mu})$
3. Interpret geometrically the Christoffel symbols of the sphere calculated in the previous problem set.
4. Calculate the transformation of the Christoffel symbols of the first kind under a general coordinate transformation and show that $A_{\mu;\nu}$ (as defined in (10.1) in D) transforms like a rank two tensor.
5. Calculate the Christoffel symbols for the two-dimensional Euclidean plane using polar coordinates. Write the geodesic equations and discuss the solutions.

Week 4

Readings: D, Chs. 15-17. See also: HEL, selected parts of Ch. 7 and 8.

Problem set 3 (due Tuesday 2/13)

1. Calculate the Euler-Lagrange equations for the Lagrangian $L = \sqrt{g_{\mu\nu}(x(\tau))\dot{x}^\mu \dot{x}^\nu}$, with $\dot{x}^\mu \equiv \frac{d}{d\tau}x^\mu(\tau)$. Construct explicitly a reparametrization $x'^\mu(\tau'(\tau)) = x^\mu(\tau)$ such that the Euler-Lagrange equations become the geodesic equations as we know them. Interpret geometrically the reparametrization.

2. Calculate the Riemann, Ricci and curvature tensor for the two dimensional sphere. Same question for the two-dimensional plane using polar coordinates.

3. Calculate the number of independent components of the Riemann tensor in 2, 3, 4 and D dimensions.

4. Show that $R^\tau_{\mu\nu\sigma;\rho} + R^\tau_{\mu\rho\nu;\sigma} + R^\tau_{\mu\sigma\rho;\nu} = 0$

5. Show that $\Gamma^\nu_{\mu\nu} = (1/2)(\ln(det g))_{,\mu}$

Week 5

Readings: D chs. 18-23. See also selected sections of chs. 9-11 in HEL.

Problem set 4 (due 2/20)

1. Calculate the Riemann, Ricci and scalar curvature R of a 3-sphere (no time). A 3-sphere can be embedded in 4 Euclidian dimensions. The 4-dimensional spherical coordinates can be constructed from the 3-dimensional ones by changing $r \to r\sin\chi$ and having $x_4 = r\cos\chi$ with $0 < \chi < \pi$.

2. Consider the metric with the invariant $c^2 d\tau^2 = f(r)c^2dt^2 - g(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$. Calculate the Christoffel symbols and the Ricci tensor for this metric. Check explicitly that your result agrees with Dirac p. 30-31.

3. Show that Kepler's problem can be reduced to $\frac{d^2u}{d\phi^2} + u$=constant. What is the constant? Give the most general solution of this equation and discuss the geometrical meaning of the constants of integration. Assuming that the total energy is strictly negative, the solutions are ellipses. Show that in this case, the solution implies that $(x/a)^2 + (y/b)^2 = 1$ with x and y the coordinates with respect to the center (and not one of the focus) of the ellipse. Interpret geometrically a and b and express them in terms of the constants of integration introduced before.

4. Write the geodesic equations with the Christoffel symbols of Schwarzschild solution. Show that they can be reduced to a Keplerian form $\frac{d^2u}{d\phi^2} + u$=constant $+ h(u)$. Give the explicit form of $h(u)$ and explain why it can be treated as a perturbation for the solar system. Calculate $\Delta\phi$/century for Mercury and Venus at first order in h.

5. Calculate $\Delta\phi$ for a light geodesic with the Christoffel symbols of Schwarzschild solution. Give a numerical estimate for $M = M_{sun}$ and the impact parameter $b = R_{sun}$.

Week 6

Readings: D. chs. 24-25; M. D. Kruskal, Phys. Rev. 119, 1743-1745 (1960) (available on-line).

Problem set 5 (due 2/27)

1. Consider the geodesic equations for the sphere (see problem set 1). Discuss the constants of motion. Show that the geodesic should be on a plane. Integrate the problem in general (without assuming a special orientation). Write a solution of the form $\phi(\theta)$.

2. Rediscuss the previous problem using the Hamiltonian formalism. Suggested readings: V. Arnold, Mathematical methods of classical mechanics, ch. 10.

3. Consider the metric $ds^2 = \frac{(d\rho)^2}{1-(\rho/r)^2} + \rho^2(d\phi)^2$ with a fixed r. Calculate the Christoffel symbols and the scalar curvature R. Discuss the solutions of the geodesic equations. Repeat the exercise with $r^2 \to -r^2$.

4. Write the invariant $c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ for the Schwarzschild metric in E-F coordinates

$$\begin{aligned} x'^0 &= ct + r + r_S \ln(r/r_s - 1) \\ x'^1 &= r \end{aligned} \qquad (1)$$

   and in Kruskal coordinates

$$\begin{aligned} x'^0 &= \sqrt{r/r_s - 1}\exp(\frac{r}{2r_S})\cosh(\frac{ct}{2r_S}) \\ x'^1 &= \sqrt{r/r_s - 1}\exp(\frac{r}{2r_S})\sinh(\frac{ct}{2r_S}) \end{aligned} \qquad (2)$$

5. Calculate the proper time elapsed between the drop of a particle at $r = r_S$ and its arrival at $r = 0$. Give a general formula. Estimate numericaly for M_{sun} and $10^{10} M_{sun}$.

Week 7

Midterm exam on 3/6

Important topics

1. Christoffel symbols, Riemann tensor etc... for simple manifolds
2. Solutions of geodesic equations
3. Covariance under general coordinate transformations
4. Equivalence principle
5. Time dilation in special and general relativity
6. Einstein equations for gravity
7. Schwarzschild solution

8. Homeworks

Week 8

Readings: D: Chs. 26-35; See also selected parts of Chs. 17-19 in HEL.

Problem set 6 (due 3/27)

1. Read a journal article reporting an experimental test of general relativity (many references can be found in Ohanian's book). Summarize the main results.

2. Discuss qualitatively all the type of solutions of the timelike and lightlike geodesics for a Schwazschild background as the initial conditions (and so the constant of integrations) are varied.

3. Solve numerically a timelike geodesic equation in a case where perturbative methods with respect to the Keplerian solution fails (but first, test your program by considering a case where perturbation works). See sample in

   http://www-hep.physics.uiowa.edu/%7Emeurice/gr/def.nb

4. Show that $\int d^4x\sqrt{-g}\ \delta R_{\mu\nu}g^{\mu\nu} = 0$, in other words that the integrand is a total derivative.

5. Calculate the Ricci tensor for a metric of the form $g_{00} = 1$, $g_{0i} = 0$ and $g_{ij} = -a(t)\delta_{ij}$ $(t = x^0)$.

Week 9

Readings: HEL Ch. 14; KT: Ch. 2.

Problem set 7 (due 4/4)

1. Draw the orbits corresponding to problem 3 of week 8.

2. HEL 18.9

3. HEL 18.19

4. HEL 14.7

5. Inserting the proper powers of k, c and $\hbar$, convert $T_0 = 2.725K$ in eV, $\rho_\gamma = \pi^2 T_0^4/15$ in eV^4 and kg/m^3, $10^{19}GeV$ in MKS units.

Week 10

Readings: HEL Ch. 15; KT: Ch. 3.

Problem set 8 (due 4/11)

1. Discuss the basic principle of ground-based laser interferometry detection of gravitational waves. For a frequency of 100 Hz, what amplitudes are expected to be detected by upgraded LIGO? Which type of sources could produce such signal?

   see K. Thorne, http://xxx.lanl.gov/pdf/gr-qc/9506086

2. Discuss the implications of the covariant conservation of the energy-momentum tensor for a perfect relativistic fluid in a R-W metric.

3. Discuss the geometric interpretation of the $k = -1$ case of the R-W metric ($k = 1$ is the 3-sphere).

4. HEL 15.2

5. HEL 15.3

Week 11

Readings: KT: Chs. 4-5.

Week 12

Midterm 2: 4/17

Week 13-14

Readings: HEL: Ch. 16 KT: Ch. 8

Final

Monday May 7, 12 noon, Room 618VAN